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Mental Models and Deductive Inferences in Mathematics Education: An Exploratory Study for Didactic Purposes

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Abstract

This article reports an exploratory study about conditional deductive inference in mathematics classes at the college level. The target was college students that had finished high school and were taking a leveling introductory course in the National University of General Sarmiento (UNGS). Course topics were a high school review. We focused here on deductive conditional reasoning because in math classes a conclusion or answer is required for the student when a mathematical property or definition is applied and generally this conclusion is derived deductively. We aimed to have guidelines for developing specific activities and situations that can promote deductive inference in mathematics classes, from a semantic point of view. To get this, we proposed to study students' responses to simple situations of deductive inference with mathematical content. We take into account the Mental Models Theory for designing the test and interpreting the responses (Johnson-Laird and Byrne, 1991, Johnson-Laird, Byrne and Shaeken, 1992, Johnson-Laird and Byrne, 2002, Johnson-Laird, 2013, Johnson-Laird and Khemlani, 2013). The Mental Model Theory (hereinafter MMT) introduces a semantic and practical perspective of reasoning. It is a cognitive theory about human processes of deduction that point to the semantic procedure of searching for interpretations of the premises to conclude a valid statement, in the context of the information given by the premises.

Introduction

This article reports an exploratory study about conditional deductive inference in mathematics classes at the college level. The target was college students that had finished high school and were taking a leveling introductory course in the National University of General Sarmiento (UNGS). The UNGS is a national public university in the Buenos Aires suburbs, at 35 km from Buenos Aires downtown city. There are technical careers, such as engineering or university technician; social careers, such as political economics, political science, public administration, educational science, etc.; pedagogical careers to become a college teacher in science, humanistic and social orientation (https://www.ungs.edu.ar/category/estudiar-en-la-ungs/carreras). At the moment that we made this study, all students were taking the initial mathematics course whose duration was 90 hours. Course topics were a high school review.

We focused on deductive conditional reasoning because in math classes a conclusion or answer is required for the student when a mathematical property or definition is applied and generally this conclusion is derived deductively. However, math teachers in the course generally do not focus on the logic involved in deductive practice. We aimed to have guidelines for developing specific activities and situations that can promote deductive inference in mathematics classes, from a semantic point of view. To get this, we proposed to study students' responses to simple situations of deductive inference with mathematical content. We administered a test with items referred to a conditional sentence on basic geometry. In each item, a question related to the antecedent or the consequent of this sentence was proposed to make appear the following four forms of basic inferences: modus ponens (MP), modus tollens (MT) and their associated fallacies: affirmation of the consequent (AC), denial of the antecedent (NA), and students must elaborate a conclusion in each item.

We take into account the *Mental Models Theory* for designing the test and interpreting the responses (Johnson-Laird and Byrne, 1991, Johnson-Laird, Byrne and Shaeken, 1992, Johnson-Laird and Byrne, 2002, Johnson-Laird, 2013, Johnson-Laird and Khemlani, 2013). The Mental Model Theory (hereinafter MMT) introduces a semantic and practical perspective of reasoning. It is a cognitive theory about human processes of deduction that point to the semantic procedure of searching for interpretations of the premises to conclude a valid statement, in the context of the information given by the premises.

The layout of this article is as follows: In section 2, we present the MMT elements that we use as a guide and an interpretation frame for the results, in section 3 we present the methodology of the study, in section 4 we analyze data, and finally, in section 5, we discuss on the analysis with a didactical perspective.

Linking Mental Models Theory with our Didactical Interest

Most mathematical handbooks of introductory algebra or analysis introduce mathematical logic, with rules and truth tables, at the beginning of the course and not referred to in posterior mathematical contexts. In contrast, we hold that the logic needed in mathematical reasoning must be taught in a mathematical context, enchained to meanings of mathematical objects and symbols. We wondered how to guide students' deductions, meanwhile they are concentrated on learning or using mathematics and not so much on which is logically correct in their conclusions or decisions. We found some useful elements to take into account in the Mental Model Theory. A mental model is a representation of what is common to a set of possibilities denoted by the word, concept, premises, etc. Johnson-Laird, who is the principal referent theorist of the Mental Model Theory, states that "the principal assumption of the theory is that individual's reason by trying to envisage the possibilities compatible with what they know or believe" (2013). He introduces a short explanation about what happens in the human deduction process: "they start with some information—perceptual observations, memories, statements, beliefs, or imagined states of affairs—and produce a novel conclusion that follows from them" (Johnson-Laird et al., 1991, pg 18). The author also states: "In many practical inferences, their starting point is a perceived state of affairs and their conclusion is a course of action. Their aim is to arrive at a valid conclusion, which is bound to be true given that their starting point is true", and then he adds "To deduce is to maintain semantic information, to simplify, and reach a new conclusion" (1991, pg. 23). The Mental Models Theory (MMT) is presented as a theory that "reconciles the semantics of truth tables with the constraints of mental processing, and that does so in a way that explains human performance" (1991, p. 43). For the MMT the mental model of a statement included in reasoning, as premise, is a possibility of occurrence of the fact declared in such a statement i.e. a possibility of this statement is true. This theory argues that reasoners consider that a statement is valid when no other information, interpretation or alternative model can refute it. The ability to construct alternative models and counterexamples is crucial to acquire deduction competence (Johnson-Laird and Khemlani, 2013). Stylianides and Stylianides (2007) point out that this theory can complement mathematics education findings' research on learning and teaching mathematical proof.

Conforming to this theory, the process deduction has the following stages or steps (Johnson Laird et al., 1992, García-Madruga, J., Gutiérrez, F., Carriedo, N., Moreno, S. and Johnson-Laird, P.,2002) : a) premises comprehension that provides mental models of this information, b) description or integration of the information from the premises, c) formulation of a putative conclusion, in this stage there is a mental model of a mechanism of getting a conclusion from the combination of premises information, throughout the connectors (and, or, then, etc.) between the premises, d) validation, search for alternative models falsifying or ratifying the conclusion.
According to the MMT (Johnson-Laird, Byrne & Schaeken, 1992; Johnson-Laird and Byrne, 2002), errors in deductive reasoning depend on limitations in:

- the working memory capacity, both in its "meanings management," which is the number of necessary mental models involved in reasoning according to premises' information and in its "temporary appearing" which is the time elapsed since when the premises introduced the aforementioned concepts;
- the extra information included by reasoners, not specified in the premises.
- keeping in working memory a large number of explicit models for connectives in propositional reasoning;
- the construction of counterexamples, either for lack of this strategy or for lack of general knowledge about the context introduced by the premises.

A counterexample is an alternative model that is consistent with the premises but not with the putative conclusion and is enough to prove that the reasoning with this conclusion is not valid (Johnson-Laird and Khemlani, 2013). Mental Models Theory has studied models of different connectives. In our work, we focus on conditional connective because we consider that it is the kernel of mathematical reasoning. Theorems, for example, are structured as: "hypothesis implies thesis". As is stated by the theory, conditional propositional connective "*p* then *q*" can be modeled with (Johnson-Laird, 1991):

Initial model: $p \quad q$

Implicit model $[p] \quad q$

...

Explicit model: Conditional

$[p] \quad [q]$

$[-p] \quad [q]$

$[-p] \quad [-q]$

"*p q*" is the initial model of the conjunction of the propositions *p* and *q*. In the explicit model, each line is an alternative model: "[*p*] [*q*]" is the explicit model of conjunction, where square brackets indicate that the set of contingencies has been exhaustively represented (Johnson-Laird, 1992).

The bi-conditional model, *if, and only if, p then q*, will be modeled as follows:

Initial model: *p q*
Explicit model: Bi-conditional
[*p*] [*q*]
[-*p*] [-*q*]

and it can be paraphrased *if p then q and if q then p.* In Johnson-Laird and Byrne (2002), with the introduction of the *core meaning* concept, the interpretation of p then q was explained as the set of possibilities in which *p is sufficient for q* and *q is necessary for p* and the tautological interpretation compatible with all possibilities. The MMT assumes that "a deduction that can be made from the initial models of the premises will be easier than one that can be made only by fleshing out the models with explicit information" (Johnson-Laird, 1992, pg. 52). In quotidian situations, speakers treat conditionals in their "defective" form and they consider that conditional is true only in the case that the antecedent occurs. This is an obstacle to construct explicit conditional alternative models.

In a conditional mathematical sentence, the antecedent (or hypothesis) is a perceived state of an abstract object or affair, and their conclusion (or thesis) not always is a course of action but a new state of affair, such as the following sentence presented in the test of our study: "If the quadrilateral is a square then its diagonals are perpendicular". Perpendicularity is a state or characteristic of the diagonals of a square. The course of action might be to draw the figure with its perpendicular diagonals. This kind of sentence is a basic indicative conditional where the antecedent describes all that one needs to know to interpret the conditional (Johnson-Laird and Byrne, 2002). Mathematical knowledge provides a framework for the antecedent and the consequent. Since the implicit model of the conditional does not specify anything about the consequent situation, when students are not used to leading with indeterminate situations, they state, forcibly, a consequent, in the wrong way.

Method

We have designed two tests with similar formatting and similar content but with different premises, using an online form. The items of the tests referred to elemental geometric content: planar geometry of triangles. Both tests were administered to randomly chosen students in each course class. The first test was a ''training'' for the second one in two directions: for students, because they were not used to this kind of task and not used to complete online forms; for researchers, it allows training to develop assessment categories and analyze the results. The tests used in this experiment were checked in a pilot experience. Before administering the first test, teachers of the different classes taught the related deduction activities, from the learning book (Carnelli et al., 2012). They also taught about simple conditional mathematical sentences in the context of the teaching subjects of the class and explained conditional structure: premises, antecedents, consequents; the semantic value of truth of conditionals, and counterexamples. We consider that this procedure of administering one similar instrument previously, gives later data less intercepted by the difficulty of filling out the item of the test and the form; when the second test was administered to the same student group, the proposed task was not so much surprising for them, and they focused on mathematical and logic subjects.

In this article, we will analyze the second test which was completed in class, individually, under teacher supervision but without teacher support. The test was administered to all the courses and all the students present that day. The time for completing the form was thirty minutes. We collected and processed the answers from 108 students. Among them, we discarded 11 because their answers were null and void in every item, so the analysis is done on 97 cases. On the responses of each item, we stated categories to correct the test. We fixed five categories to correspond them to a Likert scale. In some of them, we established sub-categories to distinguish the type of response. Our semantic content point of view has a disadvantage to fit when a response is right or not; however, it can catch nuances that are important to be considered for a didactical proposal. As we explain in the section about the tests, the tasks of the test and the categories take account of the Mental Model Theory on deduction behavior and conditional sentences. Two researchers established the categories, separately, and then discussed the results to accord them. With the established categories, two teachers made blinded corrections. In this way, we could know if the categories fit the students' responses. The assigned score in each response was with Likert values, from 1 (unsuccessful) to 5 (successful). Data analysis is explained in section 4.

Propaedeutic Actions

In this section, we briefly describe the class situation in which our tests were applied and which previous related tasks about conditional reasoning were made by students. We considered a propaedeutic situation to decrease the effect of people's memory work. Class contents were not designed for the experiment but they were of the regular course. We have chosen the class context of the Pythagorean Theorem review to apply to the tests because there was some reflection on logic in this subject in the classes about conditional structure, theorem demonstration, etc. However, the Pythagorean Theorem does not the best semantic context to study the four formatting of simple inference: modus ponens (MP), modus tollens (MT), the affirmation of the consequent (AC), and the negation of the antecedent (NA), because the direct and reciprocal conditional are both true. In all courses, the geometric subject on the Pythagorean Theorem was revised or taught in two classes of two hours each. This is a subject of secondary school. This course focuses on its calculus applications but there is also a discussion about its validation through puzzle construction and about its logical structure. Some activities proposed in the handbook (Carnelli et al., 2012), about Pythagorean Theorem, for promoting deductive reasoning were:

- Recognizing antecedent and consequent of the theorem from the following Pythagorean Theorem statement:

If BAC is a right triangle, with a right angle in A, then the lengths of the hypotenuse, named a, and of the other sides, named b and c, are such that: $a^2 = b^2 + c^2$

- Analyzing why for all right triangles is guaranteed this side relation.

- Enunciating the reciprocal conditional statement.
- Analyzing the truth value of the reciprocal theorem and its applications.
- Differentiating a conditional sentence from a biconditional sentence.
- Discussing that the truth value of a direct conditional proposition does not imply necessarily the truth value of the reciprocal one. The example discussed was:

If a quadrilateral is a square then is a rhombus

Its reciprocal is not true.

In class there were only more two activities about this logic subject and the more related to our interest was this one (Carnelli et al., pg. 81):

Activity 14

(a) What are the antecedent and the consequent of the following mathematical sentence?
The diagonals of a square are equal.
(b) If a figure meets the antecedent condition, what can you say about the figure? Why?
(c) Which is your conclusion if a figure does not meet the antecedent' condition of the item above?
(d) State the reciprocal sentence. Is the reciprocal sentence true? Why yes or why not?

With this activity, students revised the same points indicated above and constructed alternative models as counterexamples, too. The other propaedeutic action was our first test which is described below.

Designing the Tests

As we said above, we made two tests with online forms. The forms include two parts, one part with questions about the student: age, educational experience, course hour, and career; another part with reasoning questions. In the second part, a geometrical property is presented as usually appears in a bibliography or the class. This sentence is used as the first premise of the reasoning. The second premise of each item corresponds to each inference form (MP, MT, AC, NA) and is presented with a question. We take account of TMM principles and so we presented a simple situation with familiar content, we asked for a formulation and a brief explanation. We wanted to reduce the effect of the working memory, extra-information, and stimulate the construction of counterexamples.

Students must answer the question and, depending on the case, the conclusion is necessarily derived from the premises or the response can be similar to: ''no necessary conclusion can be derived''. For our didactical purpose, we propose this kind of task, where the individual is invited to elaborate an answer in a contextual affair. We have preferred it instead of a decontextualized task, with neutral conditional, or a task where individuals must judge the truth of a given conclusion because of the following reasons:

- we are interested in getting useful data in a didactical situation with instructives that can appear in the class, with specific semantic content, to elaborate a conclusion involving cognitive actions similar to these that are required in class;

- we hold that studying reasoning in a decontextualized way (Inglis & Simpson, 2009, Attridge & Inglis, 2013) is not convenient to give enough information about individual ability to deduce in a mathematical context, neither how teachers can guide students in reasoning;
- according to MMT, we assume that there is a content effect in reasoning;
- we believe that specific semantic content would be crucial in the reasoning stages stated by MMT: premises interpretations, information combination of the premises, formulation of a conclusion depending on the mental model of the premises structures, validation of the conclusion by constructing alternative mental models.

First Test

(https://docs.google.com/forms/d/1pPTb_NkzFRkD8ozSPrIVrxcJIYlWTuszhu84yUStqOY/edit?usp=sharing): The first premise in this test is: "*Diagonals of a square are perpendicular*". This sentence can be formulated as a conditional sentence: *For all plane quadrilaterals, if the quadrilateral is a square then its diagonals are perpendicular.* We expected that students would formulate the conditional sentence, without quantifier, assuming that the universe is that of the quadrilaterals, in this way: *if the quadrilateral is a square then its diagonals are perpendicular.*

The items to complete in the form are:

(a.1) Which kind of quadrilateral is referred to in the sentence?

(a.2) Enunciate the antecedent of the geometrical property presented in the sentence.

(a.3) Enunciate the consequence of the geometrical property presented in the sentence.

(a.4) Enunciate the sentence in a conditional way: if [antecedent] then [consequent].

(b.1) According to the property enunciated in (a.4), can you assure something if you know that the figure is a square? Explain briefly.

(b.2) According to the property enunciated in (a.4), can you assure something if you know that the figure is not a square? Explain briefly.

(b.3) According to the property enunciated in (a.4), can you assure something about a quadrilateral if you know that its diagonals are perpendicular? Explain briefly.

(b.4) According to the property enunciated in (a.4), can you assure something about a quadrilateral if you know that its diagonals are not perpendicular? Explain briefly.

Those instructions are related to the task described in the previous section. In items (b.1) to (b.4) we have chosen to question "can you assure something about…?", although it can seem a little ambiguous because we wanted to know if they could infer something about the other part of the property: the diagonals or the square, and because if we would answer directly, such as: "what can you assure about its diagonals?" they may answer something wrong. In this way, we would allow a response such that: "No, I can't assure anything because when diagonals are perpendicular, the quadrilateral can or cannot be a square. There are for example rhomboids, not squares that have perpendicular diagonals."

Second Test

(https://docs.google.com/forms/d/e/1FAIpQLSd6c8Ef_yGCLhBfY4MzUeDXhCdJCfc3EhSNSsuVyY2bQbsDA/viewform): The first premise in this test is: *''If a triangle has a right angle then one of its sides is longer than the other two"*. For this level course, this result can be validated by the triangle property (in any triangle, the side opposite to greater amplitude angle is longer) or by Pythagorean Theorem because using the formulae of the sides, it can be deduced numerically that the hypotenuse is longer than the other two sides.
Questions about the four formatting of direct inference are presented. The formulations are similar to these that are usually presented in class:

(a.1) Which kind of triangle is referred to in the sentence?

(a.2) Enunciate the antecedent of the geometrical property presented in the sentence.

(a.3) Enunciate the consequence of the geometrical property presented in the sentence.

(b) The following items refer to the stated property above: ''If a triangle has a right angle then one of its sides is longer than the other two"

We wanted to stimulate the use of the modus ponens (MP) with the following question:

(b.1) If you are informed that the triangle ABC has a right angle in A, according to the stated property, can you assure something of its sides? Explain briefly.

In a previous pilot experience, we noticed that if we question "what can you assure about…?" The student feels that he or she must give a response and they do not interpret that "nothing can be assured" is a possible answer, so we decided on this formulation of the question.

The question corresponding to the affirmation of the consequent (AC) is:

(b.2) If you are informed that the triangle ABC has a side longer than the other two, can you assure something of the amplitude of its angles? Explain briefly.

The question for the negation of the antecedent (NA) *is:*

(b.3) If you are informed that the triangle ABC has no right angle, according to the stated property, can you assure something of the sides of the triangle? Explain briefly.

The last statement corresponding to modus tollens (MT) is not a question but it presents options because in the pilot experience we received correct responses such that: *"if no side is longer than the other two, then all sides are equal, the triangle is equilateral and all angles measure 60°"* This response does not allow us to evaluate the modus tollens inference because individuals habitually used extra information in this response. Although the format of this statement is very different from those presented above because there is no formulation possibility, fortunately, we preserve an acceptable internal coherence in the test. To affirm this, we calculate the alpha de Cronbach whose value is: 0.77. The proposal for modus tollens (MT) is:

(b.4) If you are informed that the triangle ABC has no side longer than the other two, choose the correct answer:

- *It has a right angle.*
- *It has no right angle.*
- *I can't assure anything about the measure of its angles.*

Results

Construction of the Categories for Score Assignment

We constructed categories of assessment of the second test taking into account also the experience of the first test. We consider the following topics to assign the score:

- *Coherence.* People reason on meanings interpreted from the sentences. So, although the final response was not correct, if it was a consequence of its initial interpretation, we assign a score of 3 or 4. On the other hand, if the response was correct but not coherent with its interpretation or with the presented situation, the score was not the maximum, for example in the MP, if the triangle is a rectangle one, the correct answer is: *one side is longer than the other two*. If the initial identification of the antecedent and consequent are corresponding to the Pythagorean Theorem and the inference is based on this, we consider that the reasoner is coherent.
- *Forcing a consequence from the definition of the objects instead of the relationships presented in the implication.* In the reasoning with context, the objects involved in the premises have a definition. A word denotes a group of characteristics, and then when an object is named, usually reasoners enunciate one or more of these characteristics as consequence. There is a persistence of the definition instead of the relationship of the conditional. For example, in the MP many students derive that the triangle rectangle has an angle of 90° or that it has a hypotenuse, so they derive from the "rectangle triangle" denotation. We consider that they deduce correctly from the definition, not from the presented implication. This is a current situation which math teachers must deal with. This affair is not captured by a decontextualized test.
- *Interpretation of conditional as a biconditional*. Many students have interpreted correctly antecedent and consequent of the sentence but they are reasoning as a biconditional sentence coherently in AC and NA.

We enunciate the indicators of the categories as precisely as possible and assign scores to each one of them, so that a blinded corrector could assign scores without inconvenience. Then, in the triangulation, we compared the assignments. Over 97 cases corrected we only found three discrepancies that we solved in a discussion between correctors. We use a Likert score scale from 1 to 5, where 5 is the correct response expected. We briefly explain the categories of correction: In MP, individuals with 4 do not interpret the property correctly but reason coherently according to their interpretation. The score 3 in MP indicates that they have deduced according to the initial property but not according to their interpretation or they have deduced as a biconditional, for example, they interpreted that the antecedent is: "one side of the triangle is longer than the other two" and the consequent is "the triangle has a right angle" and when they were answered: *If you are informed that the triangle ABC has a right angle in A, according to the stated property, can you assure something of its sides? Explain briefly.* They answered that one of the sides is longer than the other two.

In AC and NA, score 3 corresponds to individuals that interpreted the conditional as a biconditional or that they extract a correct conclusion with extra information, for example, they enunciate the Pythagorean Theorem whose valid enunciate is a biconditional. In all cases, score 2 corresponds to a wrong response but with certain coherence with the context, for example, in MP, the response: "the triangle has more two angles lower than 90°" and score 1 corresponds to a wrong and unintelligible response. We exemplify the indicators for the assignment of scores 5 and 3 in Modus Ponens (see Table 1 and Table 2).

Table 1. Abilities Corresponding to Assignment of Score 5 in MP

Assignment of Score 5	
The student recognizes the structure of implication: antecedent and consequent and their contents. The interpretation of the premises content in the conditional is correct The response is correct and the student justifies with triangle property or with Pythagorean Theorem in the correct way, showing why there is a longer side or he/she fits to the data and derives according MP.	The student demonstrates a disposition to reason. He/She interprets components of the proposition, integrates information and formulates a conclusion. His/Her mental model allows him/her to formulate the correct conclusion.

Table 2. Abilities Corresponding to Assignment of Score 3 in MP

Assignment of Score 3	
The student recognizes the structure of implication $p\rightarrow q$ with their contents but from the premise p he/she derives characteristics of the right triangle, without referring to its sides, such as: "The triangle has a right angle and two acute angles." Besides the interpretation of the premises content in the conditional the student adds an extra wrong information as "The triangle has a side a side longer than the other two and the other two sides are equal" The student interprets the given property $p\rightarrow q$ as it was the Pythagorean Theorem and he/she derives from it that a side is longer than the other two.	The student demonstrates good disposition to reason. He/She interprets components of the proposition. He/She has difficulties integrating information given in the premises. In the first indicator for example, premise q is nos taken into account. It seems that the model of p-->q is [p] …. His/Her mental model allows him/her to formulate a conclusion, even if it is not correct.

Tests Processing

Descriptive Processing

As we said above, we discarded those cases without any information and rested 97 cases with some significant responses.

Characteristics of the Individuals of the Sample

In the UNGS, students live in the influence zone, they are mostly working middle class and the first generation of college students (Ezcurra, A.M., 2011). In the test, we asked the incoming students for some individual data such as: age, studies experience, which schools they come from, career preference.

Table 3 and Figure 1 summarize the age data of the individuals, divided into four groups.

Table 3. Students Ages Ranked in Four Groups

Range	17 - 20	21-23	24-27	> 28
Total: 97	64	18	5	10

We classify career preferences into *related* or *non-related* according to which major has math subject in its curriculum. For example, engineering careers, mathematics of physics teacher, economics, ecology, are *related* ones; politic sciences, history teacher, literature teacher, culture and artistic languages management, are *non-related.* In the answers, 15 (15%) students do not choose their careers and we discard them in this counting. 55 students choose related careers (57%) and 27 (28%) non-related careers.

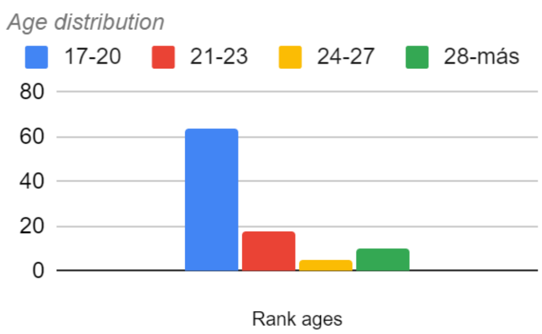

Figure 1. Four Ranks of Ages. Students are mostly in therank of [17,20] years old.

Students that are in their last year of high school could do the leveling math course. We classify the sample according to previous studies' experience: 10 students are in high school and they do not provide any information about their secondary schools. From 87 that have concluded their secondary studies, 3 of them have tertiary higher education. Of the 87 students, 41 (47%) come from public schools and 46 (52%) from private ones (Figure 2).

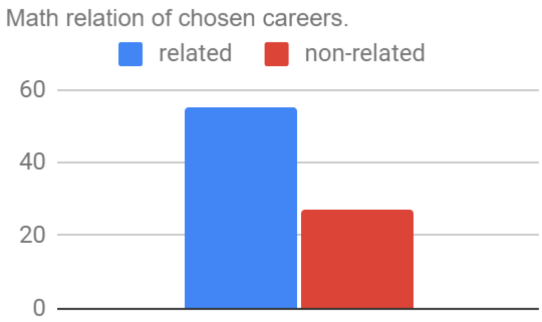

Figure 2. Counting of Students with Related or Non-related Careers

Standard Processing

In this case, students might select the correct option while in the other inferences students might formulate a conclusion. This change was introduced because in our pilot experience the responses were "all sides are equal" (or the triangle is equilateral). This response is correct but it does not correspond to the MT format and any information was taken from. The surprisingly good scores in Modus Tollens, even over the Modus Ponens scores, are probably due to the selection format of the task for this reasoning. This result indicates the difficulty of the formulation of a conclusion (Garcia Madruga et al., 2002). Since formulating a conclusion is a task very required in a class, this result also reveals that the selection's task does not show deductive ability in a complete sense. In MP, 36 individuals (37%), the sum of those with 4 and 5 scores, had reason correctly (see Table 4).

Table 4. General Scores Counting

	1	2	3	4	5
MP	18	29	14	12	24
%	19	30	14	12	25
AC	24	21	46	0	6
%	25	22	47	0	6
NA	36	22	16	7	16
%	37	23	16	7	16
MT	0	17	19	0	61
%	0	18	20	0	63

In Figure 3 and Figure 4 we summarize the scores.

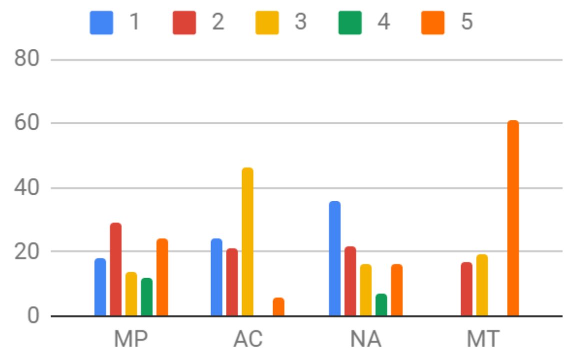

Figure 3. Graphic of general scores count

In the AC reasoning, score 3, the most frequent, is due to the biconditional interpretation of the conditional. We expected consistent responses in AC and NA of students that had interpreted the biconditional in AC but it was not so. In NA students had worse scores, many of them could not answer anything (score 1), and others gave responses referring to the type of triangle, in front of the question: *if you are informed that the triangle ABC has no right angle, according to the stated property, can you assure something of the sides of the triangle?* they answered with the evident response that the triangle was not a rectangle one.

The distribution of the averages of scores by a student is represented in Figure 4.

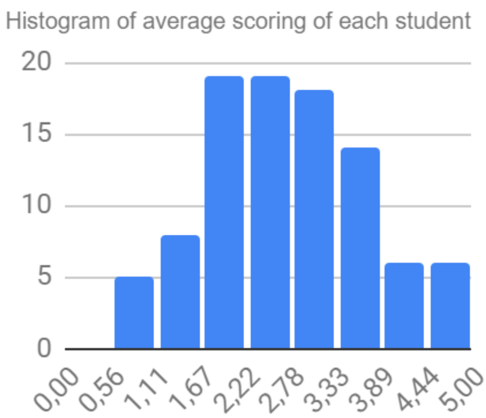

Figure 4. Histogram of Average Students' Scores in All Reasoning

The mode is 3, reached only by 12 students, the scoring average is: 2.77 and the standard deviation is 1.03. Only 14 students (14%) have scores 4 and 5. Among them, 9 choose mathematics-related careers, and two are not yet decided. As another measure of dispersion, for each inference we calculate the coefficient of variation to find out how large the corresponding deviation of scores is if we take the mean as the unit of measurement.

The averages scores and the variation coefficients by reasoning are shown in Table 5.

Table 5. Averages Scores and Variation Coefficient by Reasoning

	MP	AC	NA	MT
Average	2.95	2.41	2.43	4.08
Variation Coefficient (In percent %	50	44	60	30

We notice that the lower dispersion is in MT and the greater one is in NA. The great dispersion in the basic reasoning MP draws attention. In this reasoning, wrong responses are due to the description of the antecedent or adding extra information instead of formulating the consequent. For example, in front of the question for the MP (*Statement*: *If a triangle has a right angle then one of its sides is longer than the other two. If you are informed that the triangle ABC has a right angle in A, according to the stated property, can you assure something of its sides? Explain briefly.)* We rate with 2 responses such that:

- extra information about the sides: "the other sides are equal to one another", here the model of the triangle is an isosceles right triangle. Another response is "a^2 is the sum of b^2 and c^2", referring to the Pythagorean Theorem.
- obvious explanation or description: " I can assure that the triangle is rectangle"," b and c are the legs and a, the hypotenuse";
- particular conclusions not generalizable: "all sides are different"
- we rate with 1, nonsense responses: "a is one of the legs"
- "I don't know"
- "forced" responses: "its sides are equal by Pythagorean Theorem"

Crossing Processing

In this section we describe crossing analysis we made with the data. In all of the analyses, we apply the ANOVA test that compares means of two or more groups through a variance study in order to know if the difference is significant.

Student Age and Performance in Inferences

We analyze the crossing between students' age and their performance. We divided the population sample into four age ranges: from 17 to 20 years old, from 21 to 23, from 24 to 27 and more than 28. The initial hypothesis is that all means in the four groups are the same, the alternative one is that at least two of them are different.
In MP, all groups have mean scores greater than 2, the third group (24 to 27) has 4. In NA and AC means are lower in all groups, among 2 and 3. Finally, in MT obtained scores are better than the other inferences because, as we explained above, the task is different (selection instead of formulation of consequence) all groups have mean scores greater than 3 and the third group 4.

Although in all inferences the third group (24 to 27) has the greatest mean, the test ANOVA shows that the difference is not significant among the four age groups.

Class Schedule and Students' Performance in Inferences

Those sorts of courses are offered during the morning (8 to 12), afternoon (14 to 18) and night (18 to 22). Usually, students that work attend classes at night. So in this case we analyze the performance of three groups. In MP, the tree groups obtained a mean among 2.6 and 3.5. In NA among 2.2 and 3, in AC among 2.4 and 2.6 and in MT among 2.8 and 3.6. The lowest means corresponds to students that attend classes in the afternoon and the greatest one to students that attend classes at night. The standard error in NA is lower than in the other inferences for this group. The ANOVA test shows that there is significant difference only in NA. ANOVA does not indicate what the mean of groups is significantly different, so we make a posterior analysis post-hoc comparing the mean scores in groups of two and as result of this the third group has a significantly better performance in NA with a mean of 3.

Career and Students' Performance in Inferences

For this analysis, students that had not decided about their career were not considered. We divided the sample into two groups: Group 1: students that follow a career affin to Mathematics and Group 2: students that follow other careers. In this case, a hypothesis test in each inference is enough with the initial hypothesis that both groups had equal mean. In the two groups and the four inferences, the mean scores are between 2.25 and 3.40 and there is no significant difference between the groups. In group 1, mean scores are polarized. In AC, NA and MT, the 70% that obtained score 1 (the lowest) follow affin careers and in the four inferences the 60% that obtained score 5 (the greatest) follow affin careers.

Discussion

At the beginning of our study, we wondered about how to integrate the results of the psychological experiments with our didactical exploration. There are essential differences between these matters that make this integration difficult. First, the interpretation of a disciplinary content of the premises. Even though psychologists of the TMM hold the content effect (Johnson-Laird,1999) in reasoning, they reduce the use of significant disciplinary content in their experiments. Mathematical content is not easy to manage. Although the subjects that premises treated were affordable, this study reveals that the definition of a concept comprises components or characteristics and referring to that concept in a premise distracts attention in the working memory about the relationship between antecedent and consequent. Such is the case of the concept: *right triangle,* when you inform that there is a right triangle, reasoners of our sample focus on this definition and not on the consequence enunciated in this case: "one of sides is longer than the other two". Right triangle is an "icon" (Johnson-Laird, 2006) for reasoners and they conclude some of the parts of this icon. Since the TMM assumes that people construct a minimum of models (Johnson-Laird et al., 1991), we introduce a simple sentence related to the studied content in the course, at that moment, with the possibility of graphic construction of models. Although the initial sentence was close to the task and each item referred to it, most of the reasoners do not refer to it in the items, not even to their own initial interpretation, it seems that their working memory is erratic. In front of this difficulty, as didactical behavior, we consider that it is necessary to focus on the initial interpretation of the sentence although it seems simple and closely related to the class subject.

The *parsimonious principle* introduced by Johnson Laird et al. (2002), holds that people construct a minimum of representative models until they are forced to construct alternative ones. We expected the proposed tasks of AC and NA would force students to consider alternatives to change the initial model but most of them do not feel compelled to consider alternatives. A possible reason is that study activity in class or their handbooks, do not require this action enough and they are not habituated to this.

In the crossing analysis, contrary to what we expected, our study reveals that students with mathematics-related careers preference do not show better performance in reasoning. Respect to performance and class schedule, we observed better performance in students that attended their course at night. The group of young students, aged between 17 and 20 years who are more closely related to math contents, does not demonstrate deductive and reflective intellectual practice. The better performance of students aged 24 and over, who have finished their secondary studies five or six years ago, induce us to think that the role of secondary school pupils may be hindering, maybe individuals are habituated to play the role of student answering without reflecting or understanding. So, a recommendation as math teachers is that we should guide our students to take a reflective role as reasonners in a social community and to intensify an inference practice with usage of logical connectives with mathematical learning contents in order to argue for or against an opinion or decision.

From the report presented here, taking into account the theoretical quest of contributions of the TMM and the results of the experimentation, we extract some didactical recommendations for deductive reasoning learning in mathematics context classes. We accord with the recommendations introduced by Stylianides and Stylianides (2007) about reasoning in mathematical proving tasks: *"(1) by preventing unnecessary usage of students' working memory when they engage with proving tasks, and (2) by helping students develop strategies for effective managing of their working memory".* Our conclusions about mathematics teaching activity should be lead to introduce a practice of deductive reasoning, with short sentences referred to math content, throughout learning tasks that allow to this actions:

- to recognize in mathematical sentences the role of antecedent and consequent;
- to differentiate the definition of the mathematical object from the relationship of the object with other ones or with its characteristics. The conditions that define the object might be "encapsulated" for transcending toward the relationship with other objects or characteristics not definitories. For example, if you present the sentence: "if a figure is a square, its diagonals are perpendicular" and then, when this information is needed, students' responses refer to sides of the figure or the definition of diagonals instead of perpendicularity of diagonals.

Attending to the parsimonious principle, mathematics teaching activity should be lead to,

- distinguish between arguments that play a role of validation from those which play a role of counterexample;

- train students introducing examples and situations where the construction of counterexamples are explicitly required;
- provide different semiotic representations of mathematical objects, with activities of conversion and traduction between them, in order to enlarge the possibilities of constructing alternative models. This guideline is closely related to the Theory of the Registers of Representation Duval (1993, 2006), but the contribution of this theory would be used here not only to increase the comprehension of the object, but as a resource of constructing counterexamples.

As strategy of how to present inferences tasks, we think that they should,

- manage conditional inferences and their validity, first by selecting tasks (True or False; Always, Sometimes or Never, Yes or Not, etc.) and gradually by formulating tasks of the consequent to make the reasoning valid or the formulation of the counterexample;
- introduce short sentences with the three lines of the explicit model of the conditional to analyze each line of the model.

Educating thinking involves several intellectual components: ordering and systematization, validity analysis, knowledge and interpretation of information, reflection, imagination, awareness and putting into play different conceptions, comparisons and analogies, inferences, etc. In this sense, this study shows us how mathematics learning and teaching is a suitable space to educate thinking from a semantic perspective but the complexity of this learning requires specific activities and tasks.

References

Attridge N, Inglis M (2013) Advanced Mathematical Study and the Development of Conditional Reasoning Skills. *PLoS ONE, 8*(7): e69399. doi:10.1371/journal.pone.0069399.

Carnelli, G. Cesaratto, E., Falsetti, M., Formica, F., Marino, T. (2013) *Matemática en contexto*. Ediciones UNGS. Serie Textos Básicos. Los Polvorines, Argentina.

Duval, R (1993). Régistres de représentation sémiotique et fonctionnement cognitif de la pensée. Available in https://numerisation.irem.univ-mrs.fr/ST/IST93004/IST93004.pdf .

Duval, R. (2006) A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*. 61, 103–131.

Ezcurra, A.M., Marquina, M. Glusz, N., Feeney, S., Sosa, Y., Rosica, M., Pegasano, E. Grandoli, M.E. (2011). *Admisión a la universidad y selectividad social. Ediciones UNGS*. Los Polvorines, Argentina.

García-Madruga, J., Gutiérrez, F., Carriedo, N., Moreno, S., & Johnson-Laird, P. (2002). Mental Models in Deductive Reasoning. *The Spanish Journal of Psychology*, *5*(2), 125-140.

Inglis, M. Simpson, A., (2009). Conditional inference and advanced mathematical study: further evidence. *Educational Studies in Mathematics, 72*(2), pp.185-198.

Johnson-Laird, P. N., & Byrne, R. M. J. (1991). *Deduction*. Lawrence Erlbaum Associates, Inc.

Johnson-Laird, P. N., Byrne, R. M., & Schaeken, W. (1992). Propositional reasoning by model. *Psychological Review*, *99*(3), 418–439.

Johnson-Laird, P.N.(1999). Deductive Reasoning. *Annu. Rev. Psychol.,* 50, 109-135.

Johnson-Laird, P.N., Byrne, R. (2002) Conditionals: A Theory of Meaning, Pragmatics, and Inference. *Psychological Review*, *109*(4), 646–678.

Johnson-Laird, P. N. (2006). *How we reason.* Oxford University Press.

Johnson-Laird, P. N. (2013). Mental models and cognitive change. *Journal of Cognitive Psychology, 25*(2), 131–138.

Johnson-Laird, P.N.; Khemlani, S. (2013) Toward a Unified Theory of Reasoning. In *The Psychology of Learning and Motivation*, Academic Press, Volume 59, 1-42. http://dx.doi.org/10.1016/B978-0-12-407187-2.00001-0 1.

Stylianides, A., Stylianides, G. (2007) The mental models theory of deductive reasoning: implications for proof instruction in *Lettre de la Preuve* available in http://lettredelapreuve.org/OldPreuve/CERME5Papers/WG4-Stylianides_Stylianides.pdf.

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